NOTE

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Anharmonic oscillator and Bogoliubov transformation

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The anharmonic oscillator occupies a cornerstone in many problems in Physics. Besides its various applications, it is an interesting system in which to the studies of new effects and phenomena in quantum theory, such as, divergence of perturbation series, convergence of pade approximants, singularities in the coupling constant plane and level crossing, convergence of strong coupling expansions, perturbation theory in large order and instantons can be made. However, from the literature it is observed that none of the authors have tested Bogoliubov transformation to study anharmonic oscillator. Bogoliubov transformations are applicable to cases which satisfy commutational as well as anti-commutational relationships. An example of the later is the case of superconductivity where weak coupling regime is predominant. An example of the former is the case of superfluidity where also weak coupling regime predominates. It is therefore, evident that before testing the range of coupling constant for the applicability of Bogoliubov transformation, one must apply it to the anharmonic oscillator.

In this communication, groundstate energy of the anharmonic oscillator is studied using Bogoliubov transformation. The Hamiltonian considered here is

$$H=\frac{P^2}{2}+\frac{X^2}{2}+\lambda X^4 \tag{1}$$

Using the coordinate transformation

$$a=\frac{X+iP}{\sqrt{2}} \tag{2}$$

$$a^+=\frac{X-IP}{\sqrt{2}} \tag{3}$$

the Hamiltonian in eq. (1) is written as

$$H=a^+a+\frac{1}{2}+\frac{\lambda}{4}\Big[a^4+(a^+)^4+4(a^+)^3a+4a^+a^3+6(a^+)^2a^2+6a^2$$
$$+6(a^+)^2+12a^+a+3\Big] \tag{4}$$

The eq. (4) is written in normal order using the commutation relation.

$$[a\ a^+]=1 \tag{5}$$

Let us now pass from Bose operators a and a^+ to new Bose operators (Rumer and Ryvkin 1980) α and α^+ by means of Bogoliubov transformation

$$\begin{aligned}
&a=U\alpha-V\alpha^+\\
&a^+=U\alpha^+-V\alpha\\
&U^2-V^2=1\\
&[\alpha\ \alpha^+]=1\\
&\alpha\,|\,0\rangle=0
\end{aligned} \tag{6}$$

to ensure diagonalization of Hamiltonian in eq. (4). Before simplifying the Hamiltonian, we substitute (Rath 1984, Banerjee and Bhattacharjee 1984) the following,

$$\begin{aligned}
&a^4=-3UVa^2\\
&(a^+)^4=-3UV(a^+)^2\\
&a^+a^3=3V^2a^2\\
&(a^+)^3a=3V^2(a^+)^2\\
&(a^+)^2a^2=-\frac{UV}{2}[a^2+(a^+)^2]+2V^2a^+a.
\end{aligned} \tag{7}$$

Using the transformations in eq (7), the Hamiltonian in eq. (4) is written as

$$H=\frac{1}{2}+\frac{3\lambda}{4}+Aa^2+A(a^+)^2+Ba^+a, \tag{8}$$

where

$$A=(12V^2-6UV+6)\frac{\lambda}{4} \tag{9}$$

and

$$B=(3V^2+3)\lambda+1. \tag{10}$$

Now the eq. (8) in terms of α and α^+ can be written as

$$H=[AU^2+AV^2-BUV]\alpha^2+[AU^2+AV^2-BUV](\alpha^+)^2$$
$$+[BV^2-2AUV]\alpha\alpha^+ +[BU^2-2AUV]\alpha^+\alpha+\frac{1}{2}+\frac{3\lambda}{4} \tag{11}$$

The groundstate energy of the oscillator is

$$E=\langle 0 \mid H \mid 0 \rangle$$
$$=-2AUV+BV^2+\frac{3\lambda}{4}+\frac{1}{2} \tag{12}$$

The constant U or V is determined by making the coefficient of α^2 or $(\alpha^+)^2$ zero i.e.

$$AV^2+AU^2-BUV=0 \tag{13}$$

Further the eq. (13) is simplified using ths relation $U^2=1+V^2$ and is written as

$$(V^2)^3(18\lambda^2+12\lambda)+(V^2)^2\left(1+21\lambda+\frac{81\lambda^2}{4}\right)$$
$$+V^2\left(\frac{9\lambda^2}{4}+9\lambda+1\right)-\frac{9\lambda^2}{4}=0 \tag{14}$$

It is seen that groundstate energy calculated from the eq. (12) using the solution of V from eq. (14) yields reasonable good agreement with the previous computations and are given in Table 1.

Table I. Groundstate energy of the anharmonic oscillator.

λ	Present	Previous (Caswell 1979)	Previous (Rath 1984, Banerjee and Bhattacharjee 1984)
0.01	0.50728	0.50725	0.50733
0.1	0.56037	0.55914	0.56270
1	0.8125001	0.8037706	0.83584
10	1.88618	1.50497	1.60727

From the tabulated result, one concludes that Bogoliubov transformation is applicable to problems involing weak coupling as well as moderately strong coupling constants.

However, previously it was tested only against problems involving weak coupling constants. Apart from this, the energy expression in eq. (12) is neither variational nor perturbative in nature. This is mainly due to the transformation from non-interacting Bose operator with $a \mid 0 \rangle = 0$ to interacting Bose operator $a \mid 0 \rangle = -V \mid 1 \rangle$ by using the Bogoliubov transformation.

References

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Caswell W E 1979 *Ann. Phys.* **123** 184
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Rumer Yu B and Ryvkin M Sh 1980 *Thermodynamics and Statistical Physics and Kinetics* (Moscow : Mir Publisher) p 361